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ORIGINAL ARTICLE

The Chromatic Numbers of Knot Product Graphs

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ABSTRACT

Coloring is a process of assigning colors to a graph's vertices such that adjacent vertices are assigned distinct colors. This article contains the chromatic number and Equitable chromatic number of Knot Product graphs. Further, Edge chromatic number and Total chromatic number of Knot Product graph of path graphs are calculated.

Keywords: Knot Product graphs; Chromatic number; Equitable chromatic number; Edge chromatic number; Total chromatic number

1 INTRODUCTION

Graph coloring is one of the Prominent areas of graph theory. It is originated in the middle of 19th century. In 1852, Francis Guthrie proposed the Four Color Conjecture [1]. This conjecture states that any map on a plane can be colored by using four colors. The conjecture remained unproven for over a century. In 1976 Kenneth Appel and Wolfgang Haken gave computer-assisted proof of the Four Color Theorem [1].

Knot product graph was introduced by B. Basavanagouda and Keerthi G. Mirajkar [2]. The Knot product graph of two graphs G and H is denoted by $G \otimes H$. The Knot product graph of two graphs G and H is with vertex set $V(G) X\ V(H)$ defines as any two points (a_1, b_1) and (a_2, b_2) are adjacent whenever $a_1 = a_2$ or $[a_1$ is adjacent to a_2 and b_1 is adjacent to $b_2]$.

Let G be a simple graph having n vertices and m edges. Coloring is the process of assigning colors to a graph's vertices such that adjacent vertices are assigned distinct colors [3]. Chromatic number of a graph G is cardinality of fewest colors used to color graph G, denoted as $\chi(G)$ [3]. By Color class we mean the collection of vertices assigned with particular color. Edge Chromatic number is the fewest colors required to color edges of graph such that adjacent edges assigned distinct colors is denoted by $\chi_1(G)$ [3]. The Total chromatic number is the cardinality of smallest color set needed to color the graph's vertices, edges where no two adjacent vertices or edges share the same color and is denoted by $\chi_t(G)$. Equitable coloring is a method of coloring graph where each color classes V_i and V_j satisfy the condition $||V_i| - |V_j|| \leq 1$ [3]. The Cardinality of fewest color classes needed for Equitable coloring is Eqitable chromatic number and is denoted by $\chi_{eq}(G)$ [3]. A u-v walk of graph G is an alternating sequence between vertices and edges of G. A walk in which all the vertices are distinct is a path. Let P_n denotes the path with length n. Cycle is a non-trivial closed path. Cycle with length n is denoted by C_n. In a graph, if all the vertices are adjacent to each other then it is a Complete graph and is denoted by K_n. For undefined terminologies refer [3, 4]. To know more on coloring refer [5–11].

The following are of immediate use.

- **Remark 1.** [3] The Complete graph K_n is class one if n is even and is of class two if n is odd.
- **Remark 2.** [4] If G is a graph of class 1 then $\chi_1(G) = \triangle(G)$ and if G is a graph of class 2 then $\chi_1(G) = \triangle(G) + 1$, where $\triangle(G)$ denotes maximum vertex

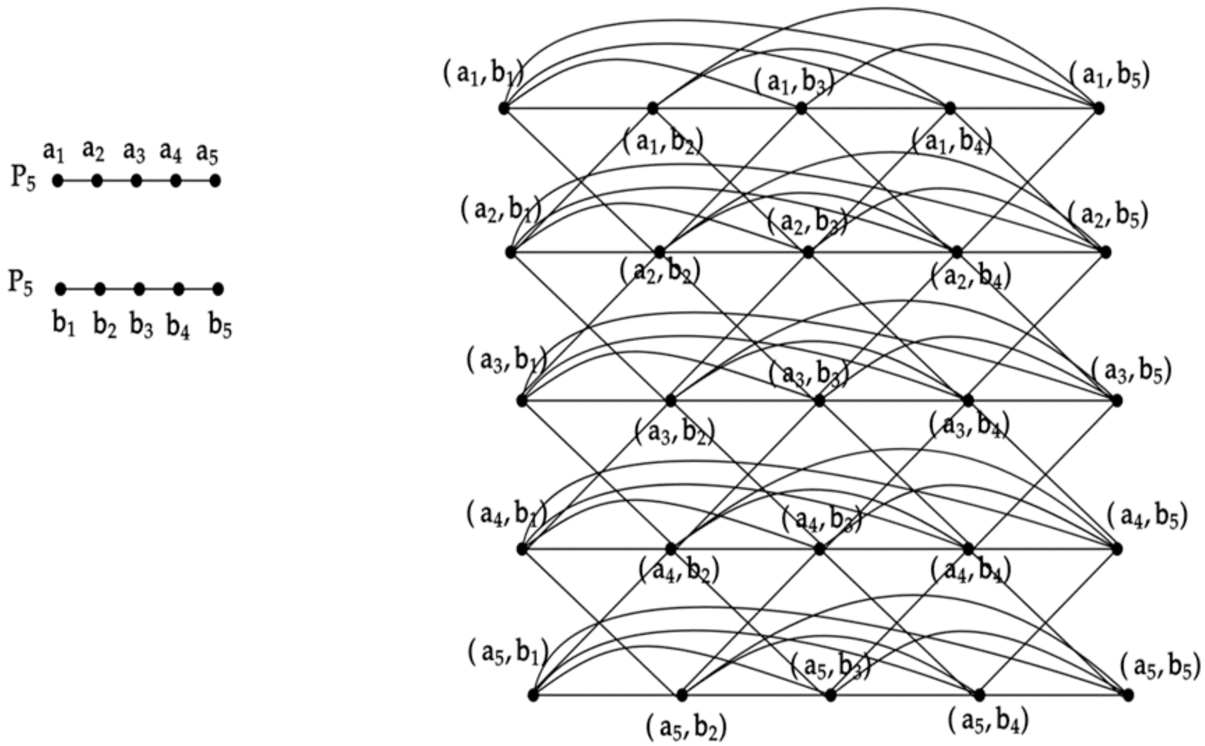

Figure 1: Knot Product graph of $P_5 \otimes P_5$

degree of G.

- **Remark 3.** [10] Total Coloring Conjecture states that, If G is class 1 graph then $\chi_t(G) = \triangle(G) + 1$and if G is a class 2 graph then $\chi_t(G) = \triangle(G) + 2$ where $\triangle(G)$ denotes maximum vertex degree of G.

2 MAIN RESULTS

This section contains chromatic number and Equitable chromatic number of Knot Product graphs. Further, Edge chromatic number and Total chromatic number of p_1 and p_n are calculated.

Theorem 2.1. The chromatic number of Knot Product graph of G_1 and G_2 is order of G_2.

Proof. Consider two graphs G_i, $i = 1, 2$ having n_i vertices and m_i edges. The Knot Product graph of G_1 and G_2 contains vertex set

$$V(G) = \{x : x = (a_i, b_i) \in V(G_1)\, X\, V(G_2)\}.$$

To compute chromatic number of Knot Product graphs consider two cases.

Case 1. $o(G_1) = o(G_2)$

Let the number of vertices in G_1 and G_2 be k = $n_1 = n_2$. Then there are 2^k count of vertices present in $V(G)$. The vertices whose y co-ordinate is v_1 are not adjacent to each other. Collection of these vertices forms independent set. The vertices whose y co-ordinate is v_2 also forms independent set. Similarly, for other vertices whose y co-ordinate is v_i also forms an independent set.

Here, $V(G)$ is seperated as k independent sets. Assign same color for vertices of a independent set. Number of colors equal to number of independent sets. Hence the chromatic number of Knot Product of graphs of G_1 and G_2 is k = $o(G_2)$.

Case 2. $o(G_1) \neq o(G_2)$

Consider G_1 and G_2 with order n_1 and n_2 respectively. The Knot Product graph of G_1 and G_2 contain $2^{n_1 n_2}$ vertices. The vertex set $V(G_1)\, X\, V(G_2)$ can be sub divided into independent sets by fixing y co-ordinate is $v_i i = 1, 2, \ldots, n_2$ and varying x co-ordinate. Each vertex set having v_i as y co-ordinate generate independent sets. The number of independent set is n_2. Each independent set receive one color. Hence, the cardinality of fewest colors required to color the graph $G_1 \otimes G_2$ is $n_2 = o(G_2)$.

By considering above two cases, it is concluded that,

$$\chi(G_1 \otimes G_2) = o(G_2).$$

Theorem 2.2. The Equitable chromatic number for Knot Product graph of G_1 and G_2 is equal to p, where p is a number of independent sets.

Proof. Consider the graphs G_1 and G_2 of order n_1 and n_2 respectively. The vertices of Knot Product graph can be grouped into p independent sets as $V_1, V_2, \ldots, V_p$ as follows.

$$V_1 = \{x : x = (a_i, b_1) \in V(G_1)\, X\, V(G_2)\}.$$

$$V_2 = \{x : x = (a_i, b_2) \in V(G_1)\, X\, V(G_2)\}.$$

$$\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{matrix}$$

$$V_p = \{x : x = (a_i, b_p) \in V(G_1)\, X\, V(G_2)\}.$$

The condition for Equitable coloring is,

$$||V_i| - |V_j|| \le 1.$$

To verify Equitable coloring consider vertex sets $V_i i = 1, 2, \ldots, p$. Each Vertex set contain n ordered pairs. Based on the information it confirms that vertex sets have same cardinality.

$$||V_i| - |V_j|| = 0 < 1.$$

From this it is concluded that graph is Equitable with respect to being p-colorable.

$$\chi_e(G_1 \otimes G_2) = |V_i|.$$

Theorem 2.3. The chromatic number of Knot Product graph of G_1 and G_2 is same as the Equitable chromatic number of Knot Product graph of G_1 and G_2.

Proof. Consider the graphs G_1 and G_2 of order n_1 and n_2 respectively.

From Theorem 2.1, chromatic number of Knot Product graph of G_1 and G_2 is,

$$\chi(G_1 \otimes G_2) = |V(G_2)| = p.$$

From Theorem 2.2, Equitable chromatic number of Knot Product graph of G_1 and G_2 is,

$$\chi_e(G_1 \otimes G_2) = p.$$

From above equations is concluded that,

$$\chi(G_1 \otimes G_2) = \chi_e(G_1 \otimes G_2).$$

Theorem 2.4. Edge Chromatic number for Knot Product graph of P_1 and P_n is

$$X_1(P \otimes P_n) = \begin{cases} n-1, & if\, n\, is\, even \\ n, & if\, n\, is\, odd \end{cases}.$$

Proof. Consider the path graph P_n, there are n vertices arranged in a linear sequence. Knot Product graph of P_1 and P_n structure contains n vertices and $\frac{n(n-1)}{2}$ edges. The Knot Product of P_1 and P_n contain vertices (a_i, b_i) which are adjacent to one other.

To calculate Edge chromatic, consider two cases.

Case 1: (If n is even)

Since the Knot Product of P_1 and P_n contain vertices (a_i, b_i) Which are adjacent to each other.

The $P_1 \otimes P_n$ Structure forms the complete graph.

From Remark 1, Complete graph is class one. From Remark 2, Edge chromatic number for class one graph is equal to maximum degree $n-1$ for complete graph when order of complete graph is even.

From Remark 1 and 2,

$$\chi_1(P_1 \otimes P_n) = n - 1.$$

Case 2: (If n is odd)

The $P_1 \otimes P_n$ Structure is a complete graph because the Knot Product of P_1 and P_n contain vertices (a_i, b_i) Which are adjacent to one other.

From Remark 1, Complete graph is class two. From Remark 2, Edge chromatic number for class two graph is equal to maximum degree n for complete graph when order of complete graph is odd.

From Remark 1 and 2,

$$\chi_1(P_1 \otimes P_n) = n.$$

Hence,

$$X_1(P \otimes P_n) = \begin{cases} n-1, & if\, n\, is\, even \\ n, & if\, n\, is\, odd \end{cases}.$$

Theorem 2.5. The Edge Chromatic number for Knot Product graph of P_n and P_1 is not Edge colorable.

Proof. Consider the path graphs P_1 and P_n. The Knot Product graph of P_n and P_1 contain n vertices. Let the vertices of Knot Product of P_n and P_1 be $(a_i, b_1) i = 1, 2, \ldots, n$. According to the Knot Product graph of G_1 and G_2 the vertices (a_i, b_i) are connected to (a_j, b_j) whenever [a_i is adjacent to a_j and b_i adjacent to b_j] or $a_i = a_j$ where $i \in n_1, j \in n_2$. The vertices of Knot Product graph of P_n and P_1 are $(a_1, b_1), (a_2, b_1), \ldots, (a_n, b_1)$. Here, $a_1 \ne a_2 \ne \ldots \ne a_n$ and b_1 can not be adjacent to b_1. Therefore, vertices of Knot Product graph of P_n and P_1 are not connected. Hence, Knot Product graph of P_n and P_1 is not Edge colorable.

Theorem 2.6. $\chi(G_1 \otimes G_2) = \chi(G_2 \otimes G_1)$ if and only if G_1 and G_2 has same number of vertices.

Proof. Consider the graphs G_1 and G_2 of order n_1 and n_2 respectively.

Let $\chi(G_1 \otimes G_2) = \chi(G_2 \otimes G_1)$. Now we have to prove G_1 and G_2 has same number of vertices.

To prove this, we consider contradiction method.

Assume that G_1 and G_2 has different order n_1 and n_2 respectively.

From Theorem 2.1,

$$\chi(G_1 \otimes G_2) = o(G_2) = n_2.$$

$$\chi(G_2 \otimes G_1) = o(G_1) = n_1.$$

Based on the above information and observation, $n_1 \neq n_2$ this implies,

$$\chi(G_1 \otimes G_2) \neq \chi(G_2 \otimes G_1).$$

Which is a contradiction. From this it is concluded that G_1 and G_2 has same number of vertices.

Conversely,

Consider, G_1 and G_2 has same number of vertices.

Now we have to prove

$$\chi(G_1 \otimes G_2) = \chi(G_2 \otimes G_1).$$

To prove this use contradiction method.

Consider,

$$\chi(G_1 \otimes G_2) \neq \chi(G_2 \otimes G_1)$$

From Theorem 2.1,

$$\chi(G_1 \otimes G_2) = o(G_2) = n_2.$$

$$\chi(G_2 \otimes G_1) = o(G_1) = n_1.$$

Based on the above information and assumption,

$$\chi(G_1 \otimes G_2) \neq \chi(G_2 \otimes G_1) \implies n_2 \neq n_1$$

Which is a contradiction.

Therefore,

$$\chi(G_1 \otimes G_2) = \chi(G_2 \otimes G_1).$$

From above information it is concluded that

$\chi(G_1 \otimes G_2) = \chi(G_2 \otimes G_1)$ if and only if G_1 and G_2 has same number of vertices.

Theorem 2.7. The Total chromatic number for Knot Product graph of P_1 and P_n is

$$X_t\ (P_1 \otimes P_n) = \begin{cases} \triangle(G)+1, & if\ n\ is\ even \\ \triangle(G)+2, & if\ n\ is\ odd \end{cases}.$$

Where $\triangle(G)$ denote maximum degree.

Proof. Consider the path graph P_n. The structure of Knot Product graph of P_1 and P_n contain n vertices and $\frac{n(n-1)}{2}$ edges. The vertices of Knot Product of P_1 and P_n are adjacent to one other.

To calculate Total Chromatic number, consider two cases.

Case 1: (If n is even)

The Structure of $P_1 \otimes P_n$ is a complete graph because from the definition of Knot Product of graph, each vertices are adjacent to each other.

From remark 1, Complete graph is class one. From remark 3, Edge chromatic number of class one graph is equal to maximum degree $\triangle(G)+1$ for complete graph when order of complete graph is even.

Hence, from remark 1 and 3,

$$\chi_1(P_1 \otimes P_n) = \triangle(G)+1.$$

Case 2: (If n is odd)

The Structure of $P_1 \otimes P_n$ is a complete graph because from the definition of Knot Product of graph all the vertices are adjacent.

From remark 1, Complete graph is class two. From remark 3, Edge chromatic number of class two graph is equal to maximum degree $\triangle(G)+2$. for complete graph when number of vertices of complete graph is odd.

From remark 1 and 3,

$$\chi_1(P_1 \otimes P_n) = \triangle(G)+2.$$

Theorem 2.8. The Total chromatic number for Knot Product graph of P_n and P_1 is equals the chromatic number for Knot Product graph of P_1 and P_n.

$$\chi_t(P_n \otimes P_1) = \chi(P_n \otimes P_1)$$

Proof. Consider the path graphs P_n and P_1. Let the vertex set of Knot Product graph of P_n and P_1 is,

$$V(G) = \{x : x = (a_i, b_i) \in V(G_1)\, X\, V(G_2)\}.$$

From the definition of Knot Product graph the vertices (a_i, b_i) are connected to (a_j, b_j) whenever [a_i is adjacent to a_j and b_i adjacent to b_j] or adjacent to $a_i = a_j$. The vertices of Knot Product graph of P_n and P_1 are $(a_1, b_1), (a_2, b_1), \ldots, (a_n, b_1)$. Here, $a_1 \neq a_2 \neq \ldots \neq a_n$ and b_1 is not adjacent to b_1. From the definition the vertices $(a_1, b_1), (a_2, b_1), (a_3, b_1), \ldots, (a_n, b_1)$ are not connected. Since, the Knot Product graph contain n vertices. Here, each vertex receives one color. To color the entire graph n colors are needed. Therefore, the total chromatic number will be n.

From Theorem 2.1,

$$\chi(P_n \otimes P_1) = n.$$

Hence,

$$\chi_t(P_n \otimes P_1) = \chi(P_n \otimes P_1).$$

3 CONCLUSION

Coloring has applications in Network Design, Resource Allocation, Scheduling problems, Frequency assignment, Chemistry. In this article the chromatic number and equitable chromatic number of Knot Product of graphs are calculated. Further, Edge chromatic number and Total chromatic number of p_1 and p_n are calculated.

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