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ORIGINAL ARTICLE

On the Bounds for Laplacian Energy of Block Adjacency Matrix

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ABSTRACT

The present research work introduces the concept of Laplacian energy with respect to block adjacency matrix named as Laplacian block adjacency energy. Further upper bound, lower bound and bound for spectral radius of Laplacian block adjacency energy are established for the graph with mutually adjacent blocks and helm graph.

Keywords: Block adjacency energy; Laplacian block adjacency energy; Spectral radius
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1 INTRODUCTION

Let G be a graph with n vertices, m edges and B blocks. Block is a maximal connected graph no cutvertices. The degree of the block in a graph G is the number of blocks adjacent to block [1]. Let d_i be the degree of the block in G. The helm graph H_t [2], where $t \geq 3$ indicates the number of pendent edges, is the graph obtained from a n-wheel graph W_n by joining a pendent edge at each vertex of the cycle. The number of blocks in H_t are $(t+1)$. Undefined graph terminologies are referred from [3]. In this paper the graphs considered are connected, simple, finite and undirected.

The Cauchy-Schwarz inequality [4] states that if $(a_1, a_2, a_3, ..., a_n)$ and $(b_1, b_2, b_3, ..., b_n)$ are real n-vectors then,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \quad (1)$$

1.1 Laplacian energy

The concept of graph energy was introduced by Gutman [5]. The adjacency matrix $A(G) = (a_{ij})$ of order n whose (i, j)-entry is defined as [5]:

$$A(G) = (a_{ij}) = \begin{cases} 1, & if\ v_i\ and\ v_j\ are\ adjacent; \\ 0, & otherwise. \end{cases}$$

If $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ of $A(G)$ are the eigenvalues of the graph G, then the spectrum of a graph G is the collection eigenvalues of adjacency matrix $A(G)$ along with their multiplicities [6]. The energy of graph $E_A(G)$ is defined to be the sum of the absolute values of eigenvalues.

$$E_A(G) = \Sigma_i^n |\lambda_i|.$$

The Laplacian matrix of G is the $n \times n$ matrix defined as $L(G) = D(G) - A(G)$, where $D(G)$ is diagonal matrix of vertex degree and $A(G)$ is adjacency matrix. This matrix has nonnegative eigenvalues $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$. The Laplacian energy of the graph [7]G is defined as

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2mn}{n}|$$

1.2 Block adjacency energy and Block adjacency Laplacian energy

For a graph G with n-vertices and $B = b_1, b_2, b_3, ...b_k; k \in N$ be the total number of blocks, $B \geq 2$, the block adjacency matrix $BA(G) = [b_{ij}]$ is defined as [8]

$$BA(G) = [b_{ij}] = \begin{cases} 1, & if\ b_i\ and\ b_j\ are\ adjacent; \\ 0, & otherwise. \end{cases}$$

The block adjacency matrix $BA(G)$ is a real symmetric matrix. If $\gamma_1, \gamma_2, \gamma_3 ..., \gamma_B$ are eigenvalues of $BA(G)$, then the block adjacency energy of a graph $E_{BA}(G)$ defined as

$$E_{BA}(G) = \sum_{i=1}^{B} |\gamma_i|$$

Graph energy and its subsequential variants have remarkable applications in various fields of science and engineering such as chemistry, crystallography, air transportation, satellite communication, face recognition, comparison of protein sequences, construction of spacecrafts, processing of high-resolution satellite images, network, and medicine [9]. Especially one of the energy variants, Laplacian graph energy found an unexpected application in image processing and classifying high resolution satellite images. The success of the theory of graph energy and its variant Laplacian energy [7, 10] inspired us to introduce the concept of Laplacian energy with respect to block adjacency matrix. Which may hold many applications in the future.

Laplacian block adjacency energy is defined as follows.

Let $D_b(G) = [d_{ij}]$ be a diagonal matrix of block degree and is defined as

$$[d_{ij}] = \begin{cases} deg(b_i), & if \ i = j \ ; \\ 0, & i \neq j \ . \end{cases}$$

The Laplacian block adjacency matrix of G is $L_{BA}(G) = D_b(G) - BA(G)$. The nonnegative eigenvalues of Laplacian block adjacency matrix are $\beta_1, \beta_2, \beta_2, ..., \beta_B$ and arranged in non increasing order $\beta_1 \geq \beta_2 \geq \beta_2 \geq, ..., \geq \beta_B$. The Laplacian block adjacency energy denoted by $LE_{BA}(G)$ is defined as

$$LE_{BA}(G) = \textstyle\sum_{i=1}^{B} |\alpha_i| \qquad (2)$$

where $\alpha_i, i = 1,2,3...,B$ is the auxillary eigenvalue. The largest eigenvalue α_1 is called the spectral radius of G.

- **For graph with mutually adjacent blocks**

  $\alpha_i = \beta_i - X(G), i = 1,2,3,...,B,$ where $X(G) = (B-1)$
.

- **For helm graph**

$\alpha_i = \beta_i - Y(G), i = 1,2,3,...,B,$ where $Y(G) = \frac{2m}{3B}$

Where B and m are the number of blocks and edges of G respectively.

In this research work Laplacian energy of block adjacency matrix for the graph with mutually adjacent blocks and helm graph are obtained. Further the bound for spectral radius, upper and lower bounds of $LE_{BA}(G)$ are obtained for the same class of graphs.

2 RESULT

The following observations are used for proving lemmas and theorems in main result.

- **Observation A.** If G be a graph with B mutually adjacent blocks, then the sum of squares of nondiagonal elements of $LE_{BA}(G) = 2\sum_{i<j} b_{ij}^2 = B(B-1)$.
- **Observation B.** If G be a graph with B mutually adjacent blocks, then the sum of diagonal elements of $LE_{BA}(G) = \sum_{i=1}^{B} d_i = B(B-1)$.
- **Observation C.** If G be helm graph with $B \geq 4$ blocks, then the sum of squares of nondiagonal elements of $LE_{BA}(G) = 2\sum_{i<j} b_{ij}^2 = 2(B-1)$.
- **Observation D.** If G be helm graph with $B \geq 4$ blocks, then the sum of the diagonal elements of $LE_{BA}(G) = \sum_{i=1}^{B} d_i = 2(B-1)$.

2.1 *Bounds on Laplacian block adjacency energy of graph with mutually adjacent blocks*

Theorem 2.1: If G be a graph with mutually adjacent $B \geq 3$ blocks, then

$LE_{BA}(G) == -2\beta_B - (B-2)X(G) - B(B-1)$

Proof. Let G be a graph with mutually adjacent $B \geq 3$ blocks and $\beta_1, \beta_2, \beta_3, ...\beta_B$ are its Laplacian block adjacency eigenvalues, where $\beta_B = 0$, $\beta_1 = \beta_{B-1} = B$.

From Equation (2), we have

$LE_{BA}(G) = \sum_{i=1}^{B} |\beta_i - X(G)|, \quad where \ X(G) = (B-1)$

$= \beta_1 + \beta_{B-1} - 2X(G) + |-(-\beta_B + X(G))| + \sum_{i=2}^{B-2} |\beta_i - X(G)|$

$= \beta_1 + \beta_{B-1} - 2X(G) - \beta_B + X(G) + \sum_{i=2}^{B-2} |\beta_i - X(G)|$

$= \beta_1 + \beta_{B-1} - \beta_B - X(G) + \sum_{i=2}^{B-2} |\beta_i - X(G)|$

$\geq \beta_1 - \beta_{B-1} - \beta_B - X(G) + |\sum_{i=2}^{B-2} [\beta_i - X(G)]|$

$= \beta_1 - \beta_{B-1} - \beta_B - X(G) + |B(B-1) - (\beta_1 + \beta_{B-1} + \beta_B) - (B-3)X(G)|$

$= \beta_1 - \beta_{B-1} - \beta_B - X(G) + B(B-1) - \beta_1 - \beta_{B-1} - \beta_B - (B-3)X(G)$

$= -2\beta_B - (B-2)X(G) - B(B-1)$

$\geq (B+1)$

The following lemmas are used in the proof of the theorems.

Lemma 2.2: If $\beta_1, \beta_2, \beta_3, ...\beta_B$ are the Laplacian block adjacency eigenvalues of graph with mutually adjacent blocks $B \geq 3$, then

$\sum_{i=1}^{B} \beta_i^2 = B(B-1) + \sum_{i=1}^{B} d_i^2$

Proof.

$\sum_{i=1}^{B} \beta_i^2 = tr(L_{BA}(G))^2 = \sum_{i=1}^{B} \sum_{j=1}^{B} b_{ij}^2$

$= 2\sum_{i<j} b_{ij}^2 + \sum_{i=1}^{B} b_{ii}^2$

By Observation A, we get

$\sum_{i=1}^{B} \beta_i^2 = B(B-1) + \sum_{i=1}^{B} d_i^2$

Lemma 2.3: If G be a graph with mutually adjacent blocks $B \geq 3$ blocks, then

$\sum_{i=1}^{B} \alpha_i = 0 \ \ and \ \ \sum_{i=1}^{B} \alpha_i^2 = S$

$where \ \ S = B(B-1) + (\sum_{i=1}^{B} d_i - (B-1))^2$

Proof. We have

$\sum_{i=1}^{B} \beta_i = tr(L_{BA}(G))$
$= \sum_{i=1}^{B} d_i$
By Observation B, we get
$= B(B-1)$.
From Equation (2) we have
$\sum_{i=1}^{B} \alpha_i = \sum_{i=1}^{B} (\beta_i - X(G))$
$= \sum_{i=1}^{B} (\beta_i - (B-1))$
$= \sum_{i=1}^{B} \beta_i - B(B-1)$
$= 0$
$\sum_{i=1}^{B} \alpha_i^2 = \sum_{i=1}^{B} (\beta_i - X(G))^2$
$= \sum_{i=1}^{B} (\beta_i - (B-1))^2$
$= \sum_{i=1}^{B} \beta_i^2 - 2(B-1)\sum_{i=1}^{B} \beta_i + (B-1)^2$
By Lemma 2.2, we get
$= B(B-1) + \sum_{i=1}^{B} d_i^2 - 2(B-1)\sum_{i=1}^{B} d_i + (B-1)^2$
$= B(B-1) + (\sum_{i=1}^{B} d_i - (B-1))^2$
$= S$
$where\ \ S = B(B-1) + (\sum_{i=1}^{B} d_i - (B-1))^2$

Theorem 2.4: If G be a graph with mutually adjacent blocks $B \geq 3$ and α_1 is the spectral radius, then
$\alpha_1 \leq \sqrt{\frac{S(B-1)}{B}}$

Proof. Let G be a graph with mutually adjacent blocks $B \geq 3$. Let $L_{BA}(G)$ be Laplacian block adjacency matrix of G and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_B$ are its eigenvalues, where α_1 is the spectral radius.

On assuming $a_i = 1$ and $b_i = \alpha_i$, $\forall i = 2, 3, ..., B$, the inequality Equation (1) becomes,

$$\left(\sum_{i=2}^{B} (1)(\alpha_i)\right)^2 \leq \left(\sum_{i=2}^{B} 1^2\right)\left(\sum_{i=2}^{B} \alpha_i^2\right) \tag{3}$$

From Lemma 2.3, we get

$$\sum_{i=1}^{B} \alpha_i = 0$$

$$\alpha_1 + \sum_{i=2}^{B} \alpha_i = 0$$

$$\left(\sum_{i=2}^{B} \alpha_i\right)^2 = (-\alpha_1)^2 \tag{4}$$

From Lemma 2.3, we get

$$\sum_{i=1}^{B} \alpha_1^2 = S$$

$$\alpha_1^2 + \sum_{i=2}^{B} \alpha_i^2 = S$$

$$\sum_{i=2}^{B} \alpha_i^2 = S - \alpha_1^2 \tag{5}$$

Now on substituting Equations (4) and (5) in Equation (3) we get
$(-\alpha_1)^2 \leq (B-1)(S - \alpha_1^2)$
$\alpha_1^2 \leq \frac{S(B-1)}{B}$
$\alpha_1 \leq \sqrt{\frac{S(B-1)}{B}}$

Theorem 2.5: If G be a graph with mutually adjacent blocks $B \geq 3$ blocks, then,
$\sqrt{S} \leq LE_{BA}(G) \leq \sqrt{BS}$

Proof. Let G be a graph with mutually adjacent blocks $B \geq 3$ blocks. Let $L_{BA}(G)$ be Laplacian block adjacency matrix of G and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_B$ are its eigenvalues.

On assuming $a_i = 1$ and $b_i = |\alpha_i|$, $\forall i = 2, 3, ..., B$, the inequality Equation (1) becomes

$$\sum_{i=1}^{B} (1)|\alpha_i|^2 \leq \left(\sum_{i=2}^{B} 1^2\right)\left(\sum_{i=2}^{B} |\alpha_i^2|\right)$$

$$\sum_{i=2}^{B} |\alpha_i|^2 \leq B\sum_{i=2}^{B} |\alpha_i^2| \tag{6}$$

From Lemma 2.3, we get

$$LE_{BA}(G) \leq \sqrt{BS} \tag{7}$$

Next consider RHS of Equation (6)

$$\left(\sum_{i=1}^{B} |\alpha_i|\right)^2 \geq \sum_{i=1}^{B} |\alpha_i|^2$$

$$LE_{BA}(G) \geq \sqrt{S} \tag{8}$$

From Equations (7) and (8) we get
$\sqrt{S} \leq LE_{BA}(G) \leq \sqrt{BS}$.

2.2 *Bounds on Laplacian block adjacency energy of helm graph*

Theorem 2.7: If G be helm graph with $B \geq 4$ blocks, then
$LE_{BA}(G) = 2\beta_1 + (B-2)Y(G) - 2(B-1)$

Proof. Let G be a helm graph with $B \geq 4$ blocks and $\beta_1, \beta_2, \beta_3, ...\beta_B$ are the Laplacian block adjacency eigenvalues of helm graph, where $\beta_B = 0$, $\beta_1 = B$, $\beta_{B-1} = 1$.

$LE_{BA}(G) = \sum_{i=1}^{B} |\beta_i - Y(G)|,\ where\ Y(G) = \frac{2m}{3B}$
$= |\beta_1 - Y(G)| + |\beta_{B-1} - Y(G)| + |\beta_B - Y(G)| + \sum_{i=2}^{B-2} |\beta_i - Y(G)|$
$= \beta_1 - Y(G) + |\beta_{B-1} + \beta_B - 2Y(G)| + \sum_{i=2}^{B-2} |\beta_i - Y(G)|$
$= \beta_1 - Y(G) + |-(-\beta_{B-1} - \beta_B + 2Y(G))| + \sum_{i=2}^{B-2} |\beta_i - Y(G)|$
$= \beta_1 - Y(G) - \beta_{B-1} - \beta_B + 2Y(G) + \sum_{i=2}^{B-2} |\beta_i - Y(G)|$
$= \beta_1 - \beta_{B-1} - \beta_B + Y(G) + \sum_{i=2}^{B-2} |\beta_i - Y(G)|$

$\geq \beta_1 - \beta_{B-1} - \beta_B + Y(G) + |\sum_{i=2}^{B} [\beta_i - Y(G)]|$
$= \beta_1 - \beta_{B-1} - \beta_B + Y(G) + |2(B-1) - (\beta_1 + \beta_{B-1} + \beta_B) - (B-3)Y(G)|$
$= \beta_1 - \beta_{B-1} + \beta_B + Y(G) + |-(-2(B-1) + \beta_1 + \beta_{B-1} + \beta_B + (B-3)Y(G))|$
$= \beta_1 - \beta_{B-1} + \beta_B + Y(G) - 2(B-1) + \beta_1 + \beta_{B-1} + \beta_B + (B-3)Y(G))$
$= 2\beta_1 + (B-2)Y(G) - 2(B-1)$
$\geq (B+1)$

The following lemmas are used in the proof of the theorems.

Lemma 2.8: If $\beta_1, \beta_2, \beta_3, ...\beta_B$ are the block adjacency eigenvalues of helm graph, $B \geq 4$, then
$\sum_{i=1}^{B} \beta_i^2 = 2(B-1) + \sum_{i=1}^{B} d_i^2$

Proof.
$\sum_{i=1}^{B} \beta_i^2 = tr(L_{BA}(G))^2$
$= \sum_{i=1}^{B} \sum_{j=1}^{B} b_{ij}^2$
$= 2\sum_{i<j} b_{ij}^2 + \sum_{i=1}^{B} b_{ii}^2$
By Observation C, we get
$\sum_{i=1}^{B} \beta_i^2 = 2(B-1) + \sum_{i=1}^{B} d_i^2$

Lemma 2.9 If G be helm graph with $B \geq 4$ blocks, then
$\sum_{i=1}^{B} \alpha_i = 0 \ \ and \ \ \sum_{i=1}^{B} \alpha_i^2 = R$
where
$R = (B-1) + 1/2(\sum_{i=1}^{B} d_i - 2m/3B)^2$

Proof. We have
$\sum_{i=1}^{B} \beta_i = tr(L_{BA}(G))$
$= \sum_{i=1}^{B} d_i$
By Observation D, we get
$= 2(B-1)$
From Equation (2), we have
$\sum_{i=1}^{B} \alpha_i = \sum_{i=1}^{B} (\beta_i - 2m/3B)$
$= \sum_{i=1}^{B} \beta_i - 2m/3$
$= 0$
$\sum_{i=1}^{B} \alpha_i^2 = \sum_{i=1}^{B} (\beta_i - 2m/3B)^2$
$= \sum_{i=1}^{B} \beta_i^2 - 4m/3B \sum_{i=1}^{B} \beta_i + 4m^2/9B^2$
By Lemma 2.8, we get
$= 2(B-1) + \sum_{i=1}^{B} d_i^2 - 4m/3B \sum_{i=1}^{B} d_i + 4m^2/9B^2$
$= 2(B-1) + (\sum_{i=1}^{B} d_i - 2m/3B)^2$
$= (B-1) + 1/2(\sum_{i=1}^{B} d_i - 2m/3B)^2$
$= R$
$where \ \ R = (B-1) + 1/2(\sum_{i=1}^{B} d_i - 2m/3B)^2$

Theorem 2.10: If G be a helm graph with blocks $B \geq 4$ and α_1 is the spectral radius, then
$\alpha_1 \leq \sqrt{\frac{R(B-1)}{B}}$

Proof. Let G be a helm graph with blocks $B \geq 4$. Let $L_{BA}(G)$ be Laplacian block adjacency matrix of G and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_B$ are its eigenvalues, where α_1 is the spectral radius.

Let $a_i = 1$ and $b_i = \alpha_i, i = 2, 3, ..., B$ then the inequality Equation (1) becomes,

$$\left(\sum_{i=2}^{B} (1)(\alpha_i)\right)^2 \leq \left(\sum_{i=2}^{B} 1^2\right)\left(\sum_{i=2}^{B} \alpha_i^2\right) \tag{9}$$

From Lemma 2.9, we get

$$\sum_{i=}^{B} \alpha_i = 0$$

$$\alpha_1 + \sum_{i=2}^{B} \alpha_i = 0$$

$$\left(\sum_{i=2}^{B} \alpha_i\right)^2 = (-\alpha_1)^2 \tag{10}$$

From Lemma 2.9, we get

$$\sum_{i=1}^{B} \alpha_1^2 = R$$

$$\alpha_1^2 + \sum_{i=2}^{B} \alpha_i^2 = R$$

$$\sum_{i=2}^{B} \alpha_i^2 = R - \alpha_1^2 \tag{11}$$

Now from Equations (9), (10) and (11) we get

$$(-\alpha_1)^2 \leq (B-1)(R - \alpha_1^2)$$

$$\alpha_1^2 \leq \frac{R(B-1)}{B}$$

$$\alpha_1 \leq \sqrt{\frac{R(B-1)}{B}}$$

Theorem 2.12: If G be a helm graph with $B \geq 4$ blocks, then,
$\sqrt{R} \leq LE_{BA}(G) \leq \sqrt{BR}$

Proof. Let G be a helm graph with $B \geq 4$ blocks. Let $L_{BA}(G)$ be Laplacian block adjacency matrix of G and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_B$ are its eigenvalues.

On assuming $a_i = 1$ and $b_i = |\alpha_i|$, $i = 2, 3, ..., B$, the inequality Equation (1) becomes

$$\sum_{i=1}^{B} (1)|\alpha_i|^2 \leq \left(\sum_{i=2}^{B} 1^2\right)\left(\sum_{i=2}^{B} |\alpha_i^2|\right)$$

$$\sum_{i=2}^{B} |\alpha_i|^2 \leq B \sum_{i=2}^{B} \alpha_i^2 \tag{12}$$

From Lemma 2.9, we get

$$LE_{BA}(G) \leq \sqrt{BR} \tag{13}$$

Now consider RHS of Equation (12)

$$\left(\sum_{i=1}^{B} |\alpha_i|\right)^2 \geq \sum_{i=1}^{B} |\alpha_i|^2$$

$$LE_{BA}(G) \geq \sqrt{R} \tag{14}$$

From Equations (12), (13) and (14) we get

$$\sqrt{R} \leq LE_{BA}(G) \leq \sqrt{BR}.$$

3 CONCLUSION

With the objective of introducing a new energy using block adjacency matrix, the present research problem is initiated and named as Laplacian block adjacency energy. The results are established for the graph with mutually adjacent blocks and helm graph. Laplacian block adjacency energy, its upper bound, lower bound and bound for spectral radius are obtained for the same class of graphs. Also, we observe that Laplacian block adjacency energy bound is $\geq (B+1)$ for both the class of graphs. Further we notice that Laplacian block adjacency energy of graph with mutually adjacent blocks is same as that of vertex adjacency energy of star graph $K_{1,m}$.

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