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Research Article

On γ-Preregular P-Open Set

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ABSTRACT

As a generalization of open sets in topological spaces, we use the notions of γ-preopen and γ-preclosed sets [1] to introduce and study the notions of γ-prp-open sets.

Keywords: γ-open; γ-preopen; γ-preclosed; γ-prp-open; γ-ppr-closed

1 INTRODUCTION

Throughout the present paper, we will denote (X,τ) as a topological space (briefly, TS). The concept of operations α on topological spaces was introduced by Kasahara [1]. Ogata [2] called the operation α as γ-operation and introduced the notion τ_γ which is the collection of all γ-open sets in a topological space. Recently, Sabir Hussain introduced and studied the properties of γ-preopen and γ-b-open sets in topological spaces.

In this paper,we introduce and explore generalized open sets namely γ-prp- open sets in topological spaces.

2 PRELIMINARIES

Definition 2.1 [3] Let (X,τ) be a TS.An operation γ on a topology τ is a mapping from τ to P(X) with N $\subseteq$ N^γ, for each N $\in$ τ, where N^γ denotes the value of γ at N. This operation is denoted by γ: $\tau \longrightarrow$ P(X).

Its complement is called γ-closed.

Definition 2.3 [2] Let N $\subseteq$ X, then γ-interior of N, denoted by int_γ (N) is defined as int_γ (N) = { p $\in$ N: p $\in$ G $\in$ τ and $G^\gamma \subseteq$ N for some G }.

Definition 2.4 [2] Let N $\subseteq$ X,then γ-closure of N,denoted by cl_γ (M) is defined as cl_γ (N) = { p $\in$ X: p $\in$ U $\in$ τ and $U^\gamma \cap$ N$\neq \phi$ for all U }.

Definition 2.5 [2] An operation γ on τ is regular if for any open nbds L,M of each p $\in$ X, there exists an open nbd N of p such that $L^\gamma \cap M^\gamma \subseteq N^\gamma$.

Definition 2.6 A set M $\subseteq$ X is called

(i) γ-preclosed [1] if cl_γ (int_γ (M))$\subseteq$M and γ-preopen if M$\subseteq$$int_\gamma$ (cl_γ (M)).

(ii) γ^*-regular-closed [4] if M=cl_γ (int_γ (M)) and γ^* -regular-open if M = int_γ (cl_γ (M)).

(iii) γ-b-closed [1] cl_γ (int_γ (M))$\cap$ int_γ (cl_γ (M))$\subseteq$M and γ-b-open if M $\subseteq$ cl_γ (int_γ (M)$\cup$ int_γ (cl_γ (M)).

The family of γ-open(resp.,γ-closed,γ-preopen,γ-preclosed, γ^*-regular open) sets of (X,τ) is denoted by τ_γ (resp.,C_γ (X), PO_γ (X), PC_γ (X), RO_{γ^*} (X)).

Definition 2.7 [1] In (X,τ),let M $\subseteq$ X. Then

(1) γ-pre-closure of M, denoted by τ_γ -pcl_γ (M) is defined as pcl_γ (M) = $\cap${U: M $\subseteq$ U and $U^c \in PO_\gamma$ (X) }.

(2) γ-pre-interior of M, denoted by $pint_\gamma$ (A) is defined as $pint_\gamma$ (M) = $\cup${H: H $\subseteq$ M and H $\in$ PO_γ (X) }.

Theorem 2.8 [1] In (X,τ),let M $\subseteq$ X. Then:

(1) pcl_γ (M) = M $\cup$ cl_γ (int_γ (M)) and $pint_\gamma$ (M) = M $\cap$ int_γ (cl_γ (M)).

(2) $pcl_\gamma(pint_\gamma(M)) = pint_\gamma(M) \cup cl_\gamma(int_\gamma(M)$ and $pint_\gamma(pcl_\gamma(M)) = pcl_\gamma(M) \cap int_\gamma(cl_\gamma(M))$.

(3) $Pint\gamma(pcl\gamma(M)) = pint\gamma(bcl\gamma(M)) = bcl\gamma(pint\gamma(M))$.

3 γ-PREREGULAR P-OPEN SETS

Definition 3.1 A subset M of a TS (X,τ) is said to be γ-preregular p- open(briefly,γ-prp-open) if $M = pint_\gamma(pcl_\gamma(M))$.

The class of γ-prp-open in (X,τ) will be denoted by $PRPO_\gamma(X)$.

Definition 3.2 *In (X,τ),let $M \sqsubseteq X$. Then M is called a γ-prp-closed set if* M^c is γ-prp-open.

Equivalently, M is called γ-prp-closed set if $M = pcl_\gamma(pint_\gamma(M))$.

The class of γ-prp-closed sets in (X,τ) will be denoted by $PRPC_\gamma(X)$.

Example 3.3 Let $\tau = \{X, \phi, \{k_1\}, \{K_2\}, \{k_1,k_2\}, \{k_1,k_3\}\}$ be a topology on $X = \{k_1,k_2,k_3\}$

For $k_2 \in \{X$ and $M \in \tau$,define $\gamma: \tau \to P(X)$ by

$$\gamma(M) = \begin{cases} M, & \text{if } k2 \in M \\ cl(M), & \text{if } k2 \notin M \end{cases}$$

Then $\tau_\gamma = \{X, \phi, \{k_1\}, \{k_1,k_2\}, \{k_1,k_3\}\}$.

$PO_\gamma(X) = \{X, \phi, \{k_1\}, \{k_2\}, \{k_1,k_2\}, \{k_1,k_3\}, \{k_2,k_3\}\}$.

$PRPO_\gamma(X) = \{X, \phi, \{k_1\}, \{k_2\}, \{k_1,k_3\}, \{k_2,k_3\}\}$.

Theorem 3.4 If M is γ-prp-open in (X,τ),then it is γ-preopen (hence γ- b-open) but not conversely.

Proof: Let M be γ-prp-open.Then $M = pint_\gamma(pcl_\gamma(M))$.

Hence $pint_\gamma(M) = pint_\gamma(pint_\gamma(pcl_\gamma(M))$
$= pint_\gamma(pcl_\gamma(M))$
$= M$.

Thus M is γ-preopen.

Example 3.5 In the Example 3.3, $\{k_1,k_2\}$ is a γ-preopen set but not a γ- prp-open set.

Theorem 3.6 If M is γ-prp-open in (X,τ),then it is γ-b-closed but not conversely.

Proof: Let M be γ-prp-open.Then
$M = pint_\gamma(pcl_\gamma(M))$.
$= bcl_\gamma(pint_\gamma(M))$ (by Theorem 2.8(iii)).
$= bcl_\gamma(M)$(by Theorem 3.4)

Thus M is γ-b-closed.

Example 3.7 In the Example 3.3, $\{k_3\}$ is a γ-b-closed set but not a γ-prp- open set.

Remark 3.8 The class of γ-prp-open sets is not closed under finite union as well as finite intersection. It will be shown in the following example.

Example 3.9 (1)In Example 3.3,let $C = \{k_1\}$ and $D = \{k_2\}$ then $C,D \in PRPO_\gamma(X)$ but $P \cup Q = \{k_1,k_2\} \notin PRPO_\gamma(X)$.

(2) In Example 3.3,let $S = \{k_1,k_3\}$ and $T = \{k_2,k_3\}$ then $S,T \in PRPO_\gamma(X)$ but $S \cap T = \{k_3\} \notin PRPO_\gamma(X)$.

Theorem 3.10 In (X,τ), let $R,S \subseteq X$, then the following results hold:

(i) If $R \subseteq S$, then $pint_\gamma(pcl_\gamma(R)) \subseteq pint_\gamma(pcl_\gamma(S))$.

(ii) If $R \in PO_\gamma(X)$, then $R \subseteq pint_\gamma(pcl_\gamma(R))$.

(iii) If $R \in PC_\gamma(X)$, then $pcl_\gamma(pint_\gamma(R)) \subseteq R$.

(iv) $pint_\gamma(pcl_\gamma(R))$ is γ-prp-open.

(v) If $R \in PC_\gamma(X)$, then $pint_\gamma(R)$ is γ-prp-open.

(vi) *If $R \in$* $PO\gamma(X)$, then $pcl\gamma(R)$ is γ-prp-open.

Proof: (i)Clear.

(ii) Since $R \subseteq pcl_\gamma(R)$ and let $R \in PO_\gamma(X)$,then $R \subseteq pint_\gamma(pcl_\gamma(R)$.

(iii) Let $R \in PC_\gamma(X)$. Since $pint_\gamma(R) \subseteq R$,then $pcl_\gamma(pint_\gamma(R) \subseteq R$.

(iv) We have

$pint_\gamma(pcl_\gamma(pint_\gamma(pcl_\gamma(R)) \subseteq pint_\gamma(pcl_\gamma(pcl_\gamma(R)) = pint_\gamma(pcl_\gamma(R))$

and $pint_\gamma(pcl_\gamma(pint_\gamma(pcl_\gamma(R))) \supseteq pint_\gamma(pint_\gamma(pcl_\gamma(A)) = pint_\gamma(pcl_\gamma(R)$. Hence $pint_\gamma(pcl_\gamma(pint_\gamma(pcl_\gamma(R))) = pint_\gamma(pcl_\gamma(R))$.

(v) Suppose that $R \in PC_\gamma(X)$. By (iii), $pint_\gamma(pcl_\gamma(pint_\gamma(R)) \subseteq pint_\gamma(R)$. On the other hand, we have $int_\gamma(R) \subseteq pcl_\gamma(pint_\gamma(R))$.

So that $pint_\gamma(R) \subseteq pint_\gamma(pcl_\gamma(pint_\gamma(R))$. Therefore $pint_\gamma(pcl_\gamma(pint_\gamma(R)) = pint_\gamma(R)$. This shows that $pint_\gamma(R)$ is γ-prp-open.

(vi) Similar to (v).

Theorem 3.11 In (X,τ), let $M \subseteq X$, the following are equivalent:

(i) M is γ-prp-open;

(ii) $M = pcl_\gamma(M) \cap int_\gamma(cl_\gamma(M))$;

(iii) $M = [M \cup cl_\gamma(int_\gamma(M))] \cap int_\gamma(cl_\gamma(M))$;

(iv) $M = pint_\gamma((bcl_\gamma(M)))$.

Proof: It follows from Theorem 2.8

Theorem 3.12 In (X,τ),let $N \subseteq X$. Then N is γ-prp-open if and only if it is γ-preopen and γ-b-closed.

Proof: From the theorems 3.4 and 3.6,we have that every γ-prp-open set is γ-preopen and γ-b-closed.

Conversely, let N be γ-preopen and γ-b-closed, then $pint_\gamma(N) = N$ and $bcl_\gamma(N) = N$.

By Theorem 2.8 (iii), $pint_\gamma(pcl_\gamma(N) = pint_\gamma(bcl_\gamma(N))$
$= pint_\gamma(N)$
$= N$.

Theorem 3.13 If N is γ-preclosed set in (X,τ),then the following are equivalent:

(1) N is γ-prp-open;

(2) N is γ-preopen.

Proof:(1) $\longrightarrow$(2):It follows from the Theorem 3.4

(2) $\longrightarrow$(1): By hypothesis and (ii),we have $pcl_\gamma(N) = N$ and $pint_\gamma(N) = N$

Then, $pint\gamma(pcl\gamma(N)) = pint\gamma(N) = N$

Theorem 3.14 If N is γ-prp-open and a γ-closed set of a space (X,τ),then N is γ-clopen.

Proof: By hypothesis, $N = pint_\gamma(pcl_\gamma(N))$ and $M = cl_\gamma(N)$.

By Theorem 2.8(ii), $N = pint_\gamma(pcl_\gamma(N))$
$= pcl_\gamma(N) \cap int_\gamma(cl_\gamma(N))$
$= pcl_\gamma(N) \cap int_\gamma(N)$
$= int_\gamma(N)$.

Therefore, N is γ-clopen.

Theorem 3.15 If N is γ-prp-open and a γ-open set in (X,τ),then N is γ^*-regular open.

Proof: Since N is γ-prp-open and γ-open, then we have $N = pint_\gamma (pcl_\gamma (N))$ and $N = int_\gamma (N)$.

By Theorem 3.11,$N = pint_\gamma (pcl_\gamma (N))$

$= [N \cup cl_\gamma (int_\gamma (N))] \cap int_\gamma (cl_\gamma (N))$

$=[N \cup cl_\gamma (N)] \cap int_\gamma (cl_\gamma (N))$

$= cl_\gamma (N) \cap int_\gamma (cl_\gamma (N))$

$= int_\gamma (cl_\gamma (N))$.

Therefore,N is γ^*-regular open

Definition 3.16 [4] In (X,τ),let $N \subseteq X$.Then N is called a γ^*-dense set if $cl_\gamma (N) = X$.

Theorem 3.17 If N is γ-prp-open and a γ^*-dense set in (X,τ),then N is

γ-preclosed.

Proof: Since N is γ-prp-open and γ^*-dense,then $N = pint_\gamma (pcl_\gamma (N))$ and $cl_\gamma (N)= X$.

By Theorem 3.11,$N = pint_\gamma (pcl_\gamma (N))$

$= pcl_\gamma (N) \cap int_\gamma (cl_\gamma (N))$

$= pcl_\gamma (N) \cap int_\gamma (X)$

$= pcl_\gamma (N) \cap X$

$= pcl_\gamma (N)$.

Therefore,N is γ-preclosed.

Definition 3.18 In (X,τ),let $N \subseteq X$.Then N is called a γ-rare set if $int_\gamma (N)= \phi$.

Theorem 3.19 If N is γ-preopen and a γ-rare set in (X,τ),then N is γ-prp-open.

Proof: Let N be γ-prp-open and γ-rare set, we have

$N = N \cap int_\gamma (cl_\gamma (N))$ and $int_\gamma (N)= \phi$.

By Theorem 2.8, $pint_\gamma (pcl_\gamma (N)) = [N \cup cl_\gamma (int_\gamma (N))] \cap int_\gamma (cl_\gamma (N))= [N \cup cl_\gamma (\phi)] \cap int_\gamma (cl_\gamma (N))$

$= [N \cup \phi] \cap int_\gamma (cl_\gamma (N))$

= N (as N is γ-preopen).

Therefore,N is γ-prp-open.

Theorem 3.20 In (X,τ), N is γ-open subset of X, then the following are equivalent:

(1) Is γ-prp-open;

(2) N is γ^*-regular open

Proof: (1) $\longrightarrow$(2):Follows from Theorem 3.15

(2) $\longrightarrow$(1):By hypothesis and (2),we have $int_\gamma (N) = N$ and $int_\gamma (cl_\gamma (N)) = N$.

By Theorem 2.8, $pint_\gamma (pcl_\gamma (N)) = [N \cup cl_\gamma (int_\gamma (N))] \cap int_\gamma (cl_\gamma (N))$

$= cl_\gamma (N) \cap N$ $pint_\gamma (pcl_\gamma (N)) = N$.

Therefore, N is γ-prp-open.

Definition 3.21 [4] In (X,τ),let $N \subseteq X$. *Then N is called γ^*-submaximal if every γ^*-dense subset of X is γ-open.*

Theorem 3.22 If every γ-preopen set is γ-open,then (X,τ) is γ^*-submaximal.

Proof: Let M be a γ-dense subset of X. Then $int_\gamma (cl_\gamma (M) = X$. So that $M \subseteq int_\gamma (cl_\gamma (M))$, M is γ-preopen.

Therefore, M is γ-open.

Theorem 3.23 In (X,τ),let γ be a regular operation on τ. Then every γ-preopen set is γ-open if and only if (X,τ) is γ^*-submaximal.

Proof: Follows from the fact that τ_γ is closed under finite intersection if γ is regular

Theorem 3.24 If (X,τ) is γ^*-submaximal and $\gamma: \tau \rightarrow P(X)$ is regular, then any finite intersection of γ-preopen sets is γ-preopen.

Proof: Follows from Theorem 3.23

Theorem 3.25 If (X,τ) is γ^*-submaximal and $\gamma: \tau \rightarrow P(X)$ is regular, then any finite intersection of γ-prp-open sets is γ-prp-open.

Proof: *Let $\{U_i:i=1,2,...,n\}$ be finite family of γ-prp-open. Since (X,τ) is γ^*-submaximal and γ is regular so that by Theorem 3.24 we have, $\bigcap_{i=n}^{n} U_i \in PO_\gamma(X)$. Therefore $\bigcap_{i=n}^{n} U_i \subseteq int_\gamma(cl_\gamma(\bigcap_{i=n}^{n} U_i)$.*

For each i, we have $\bigcap_{i=n}^{n} U_i \subseteq U_i$ and thus $pint_\gamma(pcl_\gamma(\bigcap_{i=n}^{n} U_i) \subseteq pint_\gamma(pcl_\gamma(U_i)$.

As $pint_\gamma(pcl_\gamma(U_i)=U_i$, then $pint_\gamma(pcl_\gamma(\bigcap_{i=n}^{n} U_i) \subseteq \bigcap_{i=n}^{n} U_i$.

Recall that a subset M of a TS (X,τ) is called γ-preclopen if it is γ-preclosed and γ-preopen.

Theorem 3.26 In (X,τ),let $N \subseteq X$. If N is γ-preclopen,then it is γ-prp- open but not conversely.

Proof: Obvious

Example 3.27 Let $\tau = \{X,\phi,\{p_1\}, \{p_2\}, \{p_3\}, \{p_1,p_2\},\{p_1,p_3\}, \{p_2,p_3\},\{p_1,p_2,p_3\},\{p_1,p_3,p_4\}\}$

$$\gamma(M) = \begin{cases} M\cup,\{q\}, \text{ if } M = \{p\},\{p,r,s\} \\ M, \text{ if } M \neq \{p\},\{p,r,s\} \end{cases}$$

Here $\{p_2\}$ is γ-prp-open in (X,τ) but not a γ-preclopen set

Theorem 3.28 In (X,τ),let $N \subseteq X$, *the following are equivalent:*

(i) N is preclopen;

(ii) N is γ-prp-open and γ-preclosed.

Definition 3.29 A TS (X,τ) is called extremally γ-predisconnected if the

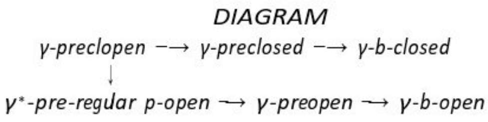
γ-preclosure of every γ-preopen subset of X is γ-preopen.

Theorem 3.30 A TS (X,τ) is extremally γ-predisconnected if and only if every γ-prp-open set is γ-preclopen.

Proof: Let M be a γ-prp-open set,then $M = pint_\gamma (pcl_\gamma (M)) = pcl_\gamma (M)$. So that M is γ-preclosed and combined with Theorem 3.4,we have M is γ-preclopen.

Conversely, let $M \in PO_\gamma (X)$.Then by Theorem 3.11(vi), $pcl_\gamma (M)$ is γ-prp- open which is γ-preclopen by hypothesis. Hence $pcl_\gamma (M)$ is γ-preopen.

Remark 3.31 The above discussions can be summarized in the following diagram:



Remark 3.32 The notions of γ-open(resp.,γ-regular open,γ-preclosed) and γ-prp-open sets are independent of each other.

Example 3.33 Let $\tau = \{X, \phi, \{a_2\}, \{a_3\}, \{a_4\}, \{a_2,a_3\},\{a_2,a_4\}, \{a_3,a_4\},\{a_1,a_3,a_4\}$, *be a topology on* $X = \{a_1,a_2,a_3 ,a_4\}$

For every{M $\in \tau$,Define γ: $\tau \to P(X)$ by

$\gamma(M) = M$ *if* $M = \{a_2\}, \{a_3\}$

$cl(M)$ *if* $M /= \{a_2\}, \{a_3\}$

Here $\{a_3\}$ is a γ-o en set in (X,τ) but $\{a_3\} \notin PRPO_\gamma$ (X) and $\{a_1,a_2,a_3\}$ is a γ-prp-o en set in (X,τ) but $\{a_1,a_2,a_3\} \notin \tau_\gamma$.

Example 3.34 In Example 3.33, $\{a_3\}$ is a γ^*-regular open set in (X,τ) but

$\{a_3\} \in/ PRPO_\gamma$ (X) and $\{a_1,a_2,a_3\}$ is a γ-prp-open set in (X,τ) but $\{a_1,a_2,a_3\} \in/ RO_\gamma{}^*$ (X).

Example 3.35 In Example 3.33, $\{a_1\}$ is a γ-preclosed set in (X,τ) but not a γ-prp-open set

Example 3.36 In Example 3.27, $\{p_2\}$ is a γ-prp-open set in (X,τ) but not a γ-preclosed set

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